Domain and Range

For a function f,

the domain is the set that is valid as input of f.

the range is the set that is the output of f.

e.g. (§ 14.1)

	function	domain	range
(1)	$\sqrt{y-x^2}$	$\sqrt{u}$: $u \geq 0$	$\sqrt{u}$: $[0, \infty)$
		$\sqrt{y-x^2}$: $y \geq x^2$	$\sqrt{y-x^2}$: $[0, \infty)$
(2).	$\frac{1}{xy}$	$\frac{1}{u}$: $u \neq 0$	$\frac{1}{u}$: $\mathbb{R} \setminus \{0\}$
		$\frac{1}{xy}$: $xy \neq 0$	$\frac{1}{xy}$: $\mathbb{R} \setminus \{0\}$
(3).	$\sin xy$	$\sin u$: $u \in \mathbb{R}$	$\sin u$: $[-1, 1]$
		$\sin xy$: $\mathbb{R}^2$	$\sin xy$: $[-1, 1]$
(4).	$\sqrt{x^2+y^2+z^2}$	$\sqrt{u}$: $u \geq 0$	$\sqrt{u}$: $[0, \infty)$
		$\sqrt{x^2+y^2+z^2}$: $\mathbb{R}^3$	$\sqrt{x^2+y^2+z^2}$: $[0, \infty)$

function	domain	range
(5) : $\frac{1}{x^2+y^2+z^2}$	$\frac{1}{u}$: $u \geq 0$	$\frac{1}{u}$: $\mathbb{R} \setminus \{0\}$
	$\frac{1}{x^2+y^2+z^2}$: $\mathbb{R}^3$	$\frac{1}{x^2+y^2+z^2}$: $(0, \infty)$
(6) $x y \ln z$	x : $\mathbb{R}$	x : $\mathbb{R}$
	y : $\mathbb{R}$	y : $\mathbb{R}$
	$\ln z$: $z > 0$	$\ln z$: $\mathbb{R}$
	$x y \ln z$: $\mathbb{R} \times \mathbb{R} \times (0,\infty)$	$x y \ln z$: $\mathbb{R}$

Limit. (§14.2)

Def. $\vec{f} : A \subseteq \mathbb{R}^n \to \mathbb{R}^m$

$\vec{f}$ is said to have a limit $\vec{L}$

as $\vec{x} \to \vec{a} \in \mathbb{R}^n$

if $\forall \varepsilon > 0$, $\exists \delta > 0$ s.t.

$\forall \vec{x} \in A$ and $0 < \| \vec{x} - \vec{a} \| < \delta$,

$\| \vec{f}(\vec{x}) - \vec{L} \| < \varepsilon$

In this case, we write

$$\lim_{\vec{x} \to \vec{a}} \vec{f}(\vec{x}) = \vec{L}.$$

Meaning: If $\vec{x}$ goes closer to $\vec{a}$,

then $\vec{f}(\vec{x})$ goes closer to $\vec{L}$.

So "a limit does not exist at $\vec{a}$" means

When $\vec{x}$ goes closer to $\vec{a}$,

$\vec{f}(\vec{x})$ will not goes to any number.

i.e.

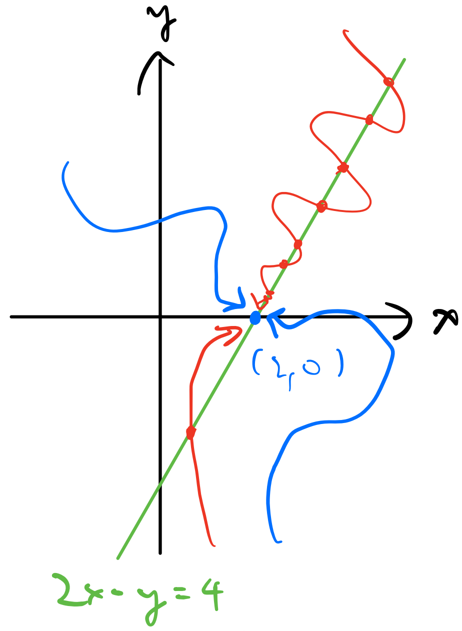
If there is two paths approaching $\vec{a}$,

that the limits are different on the two paths,

then the limit doesn't exist as $\vec{x} \to \vec{a}$.

Q 13)

$$\lim_{\substack{(x,y)\to(2,0)\\ 2x-y\neq 4}} \frac{\sqrt{2x-y}-2}{2x-y-4}$$



$$= \lim_{\substack{(x,y)\to(2,0)\\ 2x-y\neq 4}} \frac{\sqrt{2x-y}-2}{(\sqrt{2x-y}+2)(\sqrt{2x-y}-2)}$$

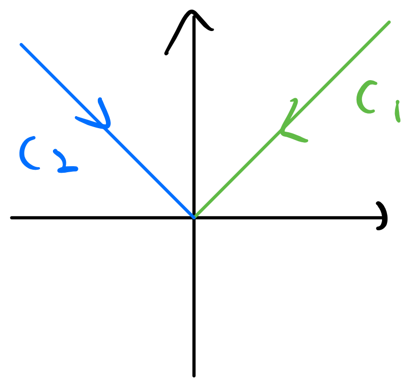
$$= \lim_{\substack{(x,y)\to(2,0)\\ 2x-y\neq 4}} \frac{1}{\sqrt{2x-y}+2} = \frac{1}{4}$$

$2x-y\neq 4$ in limit means the path chosen to approach $(2,0)$ cannot cross the line $2x-y=4$

e.g.: blue lines are OK, red lines are not.

Q 44) Show $\frac{xy}{|xy|}$ has no limit at origin.

Choose two Paths to approach origin:



C_1 : $x=y=t$, $t>0$

C_2 : $x=-t$, $y=t$, $t>0$.

$$\lim_{\substack{(x,y)\to(0,0)\\ \text{on } C_1}} \frac{xy}{|xy|} = \lim_{t\to 0} \frac{t^2}{|t^2|} = 1$$

$$\lim_{\substack{(x,y)\to(0,0)\\ \text{on } C_2}} \frac{xy}{|xy|} = \lim_{t\to 0} \frac{-t^2}{|-t^2|} = -1$$

$\therefore$ limit not exist.

Q5b)

Given :

$$2|xy| - \frac{x^2y^2}{6} < 4 - 4\cos\sqrt{|xy|} < 2|xy|$$

Find $\lim\limits_{(x,y)\to(0,0)} \frac{4-4\cos\sqrt{|xy|}}{|xy|}$

For $xy \neq 0$,

$$\lim_{(x,y)\to(0,0)} \frac{2|xy| - \frac{x^2y^2}{6}}{|xy|}$$

$$= \lim_{(x,y)\to(0,0)} \frac{2|xy| - \frac{|xy|^2}{6}}{|xy|}$$

$$= \lim_{(x,y)\to(0,0)} 2 - \frac{|xy|}{6} = 2$$

$$\lim_{(x,y)\to(0,0)} \frac{2|xy|}{|xy|} = 2$$

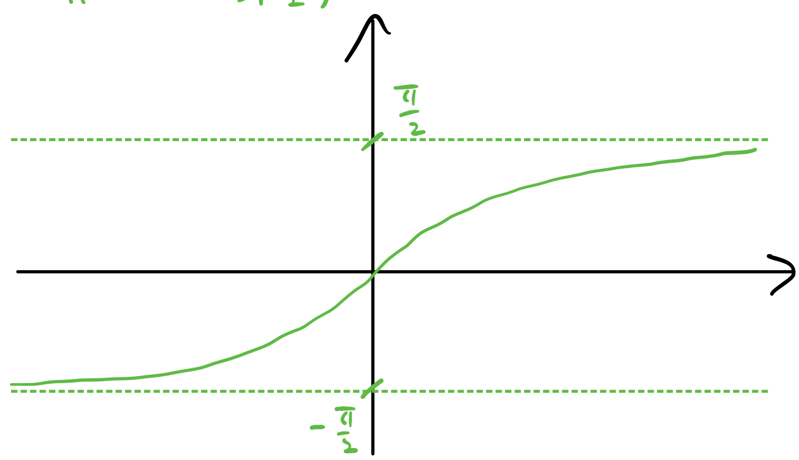
$\therefore$ Sandwich Thm

$$\Rightarrow \lim_{(x,y)\to(0,0)} \frac{4-4\cos\sqrt{|xy|}}{|xy|} = 2.$$

Q6 f). Find, or show that limit doesn't exist:

$$\lim_{(x,y)\to(0,0)} \tan^{-1}\left(\frac{|x|+|y|}{x^2+y^2}\right)$$

$\tan^{-1} : \mathbb{R} \to (-\frac{\pi}{2}, \frac{\pi}{2})$



Polar Coordinates:

$x = r\cos\theta \ , \ y = r\sin\theta$

$(x,y) \to (0,0) \iff r \to 0$

$\therefore$ Can be reduced to limit of 1 variable.

$$\lim_{(x,y)\to(0,0)} \tan^{-1}\left(\frac{|x|+|y|}{x^2+y^2}\right)$$

$$= \lim_{r\to 0} \tan^{-1}\left(\frac{|r\cos\theta|+|r\sin\theta|}{r^2\cos^2\theta+r^2\sin^2\theta}\right)$$

$$= \lim_{r\to 0} \tan^{-1}\left(\frac{|\cos\theta|+|\sin\theta|}{|r|}\right)$$

$$1 \le |\cos\theta| + |\sin\theta| \le \sqrt{2} \quad \text{——} \quad (*)$$

$$\Rightarrow \tan^{-1}\left(\frac{1}{|r|}\right) \le \tan^{-1}\left(\frac{|\cos\theta|+|\sin\theta|}{|r|}\right) \le \tan^{-1}\left(\frac{\sqrt{2}}{|r|}\right)$$

$$r \to 0^+ \Rightarrow \frac{1}{|r|}, \frac{\sqrt{2}}{|r|} \to +\infty$$

$$\Rightarrow \tan^{-1}\left(\frac{1}{|r|}\right), \tan^{-1}\left(\frac{\sqrt{2}}{|r|}\right) \to \frac{\pi}{2}$$

$$r \to 0^- \Rightarrow \frac{1}{|r|}, \frac{\sqrt{2}}{|r|} \to +\infty$$

$$\Rightarrow \tan^{-1}\left(\frac{1}{|r|}\right), \tan^{-1}\left(\frac{\sqrt{2}}{|r|}\right) \to \frac{\pi}{2}$$

Sandwich Thm

$$\Rightarrow \tan^{-1}\left(\frac{|\cos\theta|+|\sin\theta|}{|r|}\right) \to \frac{\pi}{2} \text{ as } r \to 0$$

$$\therefore \lim_{(x,y)\to(0,0)} \tan^{-1}\left(\frac{|x|+|y|}{x^2+y^2}\right) = \frac{\pi}{2}$$

(*):

$$\left(|\sin\theta| + |\cos\theta|\right)^2$$

$$= \sin^2\theta + \cos^2\theta + 2|\sin\theta\cos\theta|$$

$$= 1 + 2|\sin\theta\cos\theta|$$

$$= 1 + \underbrace{|\sin 2\theta|}_{\in [0,1]}$$

$$\therefore\ 1 \le \big(\underbrace{|\sin\theta| + |\cos\theta|}_{\text{always positive}}\big)^2 \le 2$$

$$\Rightarrow\ 1 \le |\sin\theta| + |\cos\theta| \le \sqrt{2}.$$